DEGREE OF THE FIRST INTEGRAL OF A FOLIATION IN THE PENCIL $\mathcal{P}_4$

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Abstract

Let $\mathcal{P}_4$ be the linear family of foliations of degree 4 in $\mathbb{P}^2$ given by A. Lins Neto, whose set of parameters with first integral $I_p(\mathcal{P}_4)$ is dense and countable. In this work, we will calculate explicitly the degree of the rational first integral of the foliations in this linear family, as a function of the parameter.

1 Introduction

One of the main problems in the theory of planar vector fields is to characterize the ones which admit a first integral. The invariant algebraic curves are a central object in integrability theory since 1878, year when Darboux found connections between algebraic curves and the existence of first integrals of polynomial vector fields. Thus, the first question was to know if a polynomial vector field has or not invariant algebraic curves, which was partially answered by Darboux in [5]. The most important improvements of Darboux's results were given by Poincaré in 1891, who tried to answer the following question:

"Is it possible to decide if a foliation in $\mathbb{P}^2$ has a rational first integral?"

This problem is known as the *Poincaré Problem*. In [11], he observed that it is sufficient to bound the degree of a possible algebraic solution. By imposing conditions on the singularities of the foliation he obtains necessary conditions which guarantee the existence of a rational first integral. More recently, this problem has been reformulated as follows: given a foliation on $\mathbb{P}^2$, try to bound the degree of the generic solution using information depending only on the foliation, for example its degree or the eingenvalues of its singularities.

Several authors studied this problem, see for instance [2, 3, 6, 13]. In 2002, Lins Neto (cf. [9]) built some notable 1-parameter families of foliations in $\mathbb{P}^2$, where the set of parameters in which the foliation has a first integral is dense and countable. The importance of these families is that there is no bound depending only on the degree and the analytic type of their singularities. One of such families is the pencil $\mathcal{P}_4$ in $\mathbb{P}^2$, whose set of parameters of foliations which have a first integral, denoted by $I_p(\mathcal{P}_4)$, is the imaginary quadratic field $\mathbb{Q}(\tau_0)$, where $\tau_0 = e^{2\pi i/3}$.

The purpose of this work is to calculate the degree of the foliations in $\mathcal{P}_4$ with rational first integral as a function of the parameter. For this, we first relate the pencil $\mathcal{P}_4$ with a pencil of linear foliations $\mathcal{P}_4^*$ in a complex torus $E \times E$, where $E = \mathbb{C}/\langle 1, \tau_0 \rangle$. Then we derive the formula of the degree using the ideal norm of the ring $\mathbb{Z}[\tau_0]$ as sketched below. Consequently, we are capable to address the Poincaré Problem for the foliations in $\mathcal{P}_4$.

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Given a foliation $\mathcal{F}_t \in \mathcal{P}_4$, with $t \in I_p(\mathcal{P}_4)$ there exists an unique foliation $\mathcal{G}_{\alpha(t)} \in \mathcal{P}_4^*$ where $\alpha(t) = \dfrac{t-1}{-2-\tau_0}$. Then writing $\alpha(t) = \dfrac{\alpha_1}{\beta_1}$, with $\alpha_1, \beta_1 \in \mathbb{Z}[\tau_0]$ and $(\alpha_1, \beta_1) = 1$, we have proved the following result:

Theorem. *If d_t is the degree of the first integral of $\mathcal{F}_t^4$ then*

$$d_t = N(\beta_1) + N(\alpha_1) + N(\beta_1 - \alpha_1) + N(\beta_1 + \tau_0\alpha_1),$$

where $N(\beta) = a^2 + b^2 - ab$, for $\beta = a + \tau_o b \in \mathbb{Z}[\tau_0]$.

Besides we compute the growth of the function which associates to every $n \in \mathbb{N}$, the number of parameters for which the corresponding foliation has a first integral of degree at most n. More specifically, if $\pi_{\mathcal{P}_4}(n)$ denote the number of parameters with first integral of degree at most $n \in \mathbb{N}$, then

$$\pi_{\mathcal{P}_4}(n) = O(n^2).$$

2 Preliminaries

Let $K \subset \mathbb{C}$ an *algebraic number field* and $\mathcal{O}_K$ the ring of algebraic integers contained in K. Given an ideal I of $\mathcal{O}_K$ we consider the quotient ring $\mathcal{O}_K/I$ which is finite (cf. [14, p. 106]). The *ideal norm* of I, denoted by $N_{\mathcal{O}_K}(I)$, is the cardinality of the $\mathcal{O}_K/I$.

The *Dedekind Zeta Function of K* is defined for a complex number s with $Re(s) > 1$, by the Dirichlet series

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} \frac{1}{N_{\mathcal{O}_K}(I)^s},$$

where I ranges through the non-zero ideals of the ring of integers $\mathcal{O}_K$ of K. This sum converges absolutely for all complex numbers s with $Re(s) > 1$. Note that $\zeta_{\mathbb{Q}}$ coincides with the Riemann zeta function.

Let $E = \mathbb{C}/\Gamma$ be an elliptic curve, where $\Gamma = \langle 1, \tau \rangle$ and $\mathrm{End}(E) := \mathrm{Hom}(E, E)$. Then the field $\mathrm{End}(E) \otimes \mathbb{Q}$ is isomorphic to a number field K such that $\mathcal{O}_K \simeq \mathrm{End}(E)$. Let $\alpha, \beta \in \mathrm{End}(E)$, then define the morphism $\varphi_{\alpha,\beta} : E \to E \times E$ as

$$\varphi_{\alpha,\beta}(x) = (\alpha x, \beta x).$$

Note that the image $E_{\alpha,\beta}$ of $\varphi_{\alpha,\beta}$ is an elliptic curve. Given $\alpha, \beta, \gamma, \delta \in \mathrm{End}(E)$, then the *intersection number* of the elliptic curves $E_{\alpha,\beta}$ and $E_{\gamma,\delta}$ is given by

$$E_{\alpha,\beta} \cdot E_{\gamma,\delta} = \frac{N_{\mathfrak{O}_K}\left(\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)}{N_{\mathfrak{O}_K}(\alpha,\beta) N_{\mathfrak{O}_K}(\gamma,\delta)}, \tag{1}$$

where $N_{\mathcal{O}_K}(a_1, \ldots, a_r)$ is the norm of the ideal generated by $a_1, \ldots, a_r \in \mathrm{End}(E)$ (cf. [8, Lemma 3]).

As an application consider the following example:

Example 1. Let the elliptic curve $E = \mathbb{C}/\langle 1, \tau_0 \rangle$, with $\tau_0 = e^{2\pi i/3}$, then $\mathrm{End}(E) \simeq \mathbb{Z}[\tau_0]$. Given $\alpha = a + \tau_0 b \in \mathbb{Z}[\tau_0]$ the norm of ideal $\langle \alpha \rangle$ is $N_{\mathbb{Z}[\tau_0]}(\alpha) = |\alpha|^2 = a^2 + b^2 - ab$. By (1), given $\alpha, \beta, \gamma, \delta \in \mathbb{Z}[\tau_0]$ such that $(\alpha, \beta) = 1$ and $(\gamma, \delta) = 1$ the intersection number of the elliptic curves $E_{\alpha,\beta}$ and $E_{\gamma,\delta}$ is

$$E_{\alpha,\beta} \cdot E_{\gamma,\delta} = N_{\mathbb{Z}_{\tau_0}}(\alpha\gamma - \beta\delta). \tag{2}$$

From now on, τ_0 will denote the complex number $e^{2\pi i/3}$.

2.1 The pencil $\mathcal{P}_4$ in $\mathbb{P}^2$ and the configuration $\mathcal{C}$

In [9, §2.2], Lins Neto defines the pencil $\mathcal{P}_4 = \{\mathcal{F}_\alpha^4\}_{\alpha\in\overline{\mathbb{C}}}$ of degree 4 in $\mathbb{P}^2$, where $\mathcal{F}_\alpha^4$ is defined by the 1-form $\omega + \alpha\eta$, where

$$\omega = (x^3-1)xdy - (y^3-1)ydx,$$
$$\eta = (x^3-1)y^2dy - (y^3-1)x^2dx,$$

Let us state some properties of the pencil $\mathcal{P}_4$:

1. The tangency set of the pencil $\mathcal{P}_4$, given by $\omega \wedge \eta = 0$, is the algebraic curve

$$\Delta(\mathcal{P}_4) = \big\{[x:y:z] \in \mathbb{P}^2 \;:\; (x^3-z^3)(y^3-z^3)(x^3-y^3) = 0\big\}.$$

   Then $\Delta(\mathcal{P}_4)$ is formed by nine invariant lines. Besides, the set of intersections of these lines is formed by twelve points. We will denote such lines and points by $\mathcal{L} = \{L_1,\dots,L_9\}$ and $P = \{e_1,\dots,e_{12}\}$.

2. If $\alpha \notin \{1,\tau_0,\tau_0^2,\infty\}$ then $\mathcal{F}_\alpha$ has 21 non-degenerated singularities, where nine of them are of type $(-3:1)$, and the remaining twelve are radial singularities contained in P. In particular, $\mathcal{F}_\alpha$ has degree 4.

3. If $\alpha \in \{1,\tau_0,\tau_0^2,\infty\}$ then $\mathrm{Sing}(\mathcal{F}_\alpha) = P$.

Let $\mathcal{C} = \{\mathcal{L}, P\}$ be the configuration of points and the nine lines in $\mathbb{P}^2$, as showed in Figure 1.

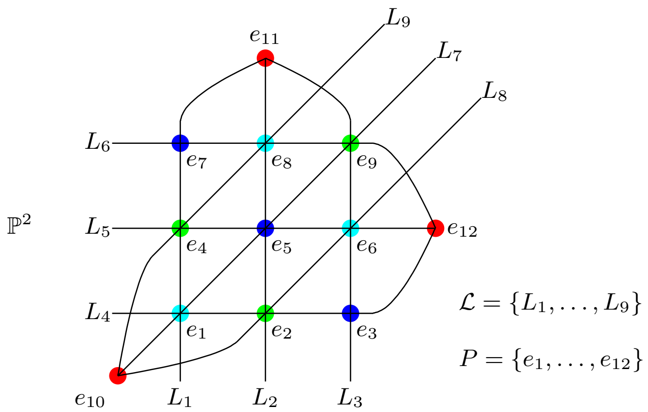

Figure 1

2.2 The pencil $\mathcal{P}_4^*$

Let $E = \mathbb{C}/\Gamma$ be an elliptic curve, where $\Gamma = \langle 1,\tau\rangle$ and $X = E \times E$. Let (x,y) be a system of coordinates of $\mathbb{C}^2$ and $\pi : \mathbb{C}^2 \to X$ be the natural projection. Let $\mathcal{P}_1 = \{\mathcal{F}_\alpha\}_{\alpha\in\overline{\mathbb{C}}}$ be the pencil of linear foliations in $\mathbb{C}^2$, where $\mathcal{F}_\alpha$ is induced by the 1-form

$$\omega_\alpha = dy - \alpha dx. \tag{3}$$

Then, using π, we obtain a pencil of linear foliations $\mathcal{P} = \{\mathcal{G}_\alpha\}_{\alpha\in\overline{\mathbb{C}}}$ in X. Define

$$I_p(\mathcal{P}) := \{\alpha \in \overline{\mathbb{C}} \;:\; \mathcal{G}_\alpha \text{ has an holomorphic first integral}\}.$$

Given $\alpha \in \mathbb{C} \setminus \{0\}$, let $L_\alpha = \{(\pi(x), \pi(\alpha x)) \ : \ x \in \mathbb{C}\}$ be the leaf of $\mathcal{G}_\alpha$ passing though $(0,0)$. Then:

$$\begin{aligned}\#\big(L_\alpha \cap (\{0\} \times E)\big) < \infty &\iff \exists\, k \in \mathbb{N} \ : \ k\alpha(m + \tau n) \in \Gamma, \forall\, m, n \in \mathbb{Z}, \\ &\iff \exists\, k \in \mathbb{N} \ : \ k\Gamma(\alpha) \subset \Gamma, \text{ where } \Gamma(\alpha) = \alpha\Gamma.\end{aligned}$$

In particular, for $\alpha \in \mathbb{C} \setminus \{0\}$, $\mathcal{G}_\alpha$ has an holomorphic first integral if, and only if, there exists $k \in \mathbb{N}$ such that $k\Gamma(\alpha) \subset \Gamma$. So we have the following Lemma.

Lemma 2. *Let $\mathcal{P} = \{\mathcal{G}_\alpha\}_{\alpha \in \overline{\mathbb{C}}}$ be a pencil of linear foliations in X, as above. Then*

$$I_p(\mathcal{P}) = (\mathbb{Q} + \tau\mathbb{Q}) \cup \{\infty\}.$$

In the case $\Gamma_0 = \langle 1, \tau_0 \rangle$ and $E_0 = \mathbb{C}/\Gamma_0$, denoted $X_0 = E_0 \times E_0$. The pencil $\{\mathcal{G}_\alpha\}_{\alpha \in \overline{\mathbb{C}}}$ in X_0 induced by (3) will be denoted by $\mathcal{P}_4^*$. In particular, by Lemma 2 we have

$$I_p(\mathcal{P}_4^*) = (\mathbb{Q} + \tau_0\mathbb{Q}) \cup \{\infty\} = \mathbb{Q}(\tau_0) \cup \{\infty\}. \tag{4}$$

2.3 The configuration $\mathcal{C}^*$ in X_0

Let $\varphi : X_0 \to X_0$ be the holomorphic map defined by $\varphi(x, y) = (\tau_0 x, \tau_0 y)$. Then,

1. $\varphi^3 = id_X$.
2. Defining $p_1 = 0$, $p_2 = \frac{2}{3} + \frac{1}{3}\tau_0$ and $p_3 = \frac{1}{3} + \frac{2}{3}\tau_0$ then $\mathrm{Fix}(\varphi) = \{(p_l, p_k)\}_{l,k=1}^{3}$ is the set of the nine fixed points of φ. Denote by $\{l_k\}_{k=1}^{9}$ the nine fixed points of φ, then

$$\mathrm{Fix}(\varphi) = \{l_1, \ldots, l_9\}.$$

Now consider the four elliptic curves in X_0:

$$\begin{aligned} E_{0,1} &= \{0\} \times E_0, & E_{1,1} &= \{(x, x) \ : \ x \in E_0\}, \\ E_{1,0} &= E_0 \times \{0\}, & E_{1,-\tau_0} &= \{(x, -\tau_0 x) \ : \ x \in E_0\}. \end{aligned}$$

Let $\mathscr{C}$ the set of these four elliptic curves. Given $F \in \mathscr{C}$ and $p \in \mathrm{Fix}(\varphi)$, denote $F_p = F + p$. Hence, the set $\mathcal{E} := \{F_p \ : \ p \in \mathrm{Fix}(\varphi), F \in \mathscr{C}\}$ consists of twelve elliptic curves, which we denote $E_1, \ldots, E_{12}$, that is,

$$\mathcal{E} = \{E_1, \ldots, E_{12}\}.$$

Since, $\varphi(F_p) = F_p$ and $\mathrm{Fix}(\varphi) \cap F_p = (\mathrm{Fix}(\varphi) \cap F) + p$ then fixed two different elliptic curves they intersect only in three fixed points of φ.

Let $\mathcal{C}^* = \big(\mathrm{Fix}(\varphi), \mathcal{E}\big)$ be the configuration of points and elliptic curves in X_0, showed in Figure 2.

3 Relation between the pencils $\mathcal{P}_4^*$ and $\mathcal{P}_4$

The relation between the pencils $\mathcal{P}_4^*$ and $\mathcal{P}_4$ was given by McQuillan in [1, p. 108], where he proved the existence of a rational map $g : X_0 \dashrightarrow \mathbb{P}^2$ such that $g^*(\mathcal{P}_4) = \mathcal{P}_4^*$. We now give an idea of how the function g is constructed. We refer the reader to [12] for the details.

Let $\pi : Bl_{\mathrm{Fix}(\varphi)}(X_0) \to X_0$ be obtained from X_0 by blowing-up the nine fixed points of φ, and denote $D_k = \pi^{-1}(l_k)$, for $k = 1, \ldots, 9$. Defining $\widetilde{X} = Bl_{\mathrm{Fix}(\varphi)}(X_0)$, there is an automorphism

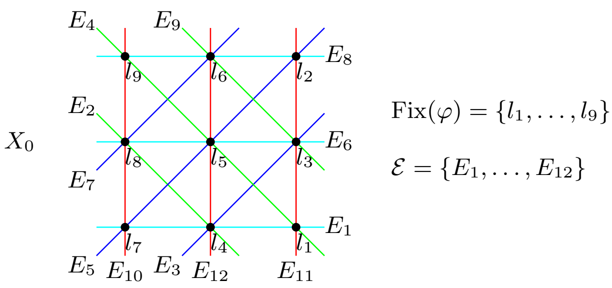

Figure 2

$\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}$ such that $\pi \circ \widetilde{\varphi} = \varphi \circ \pi$. Let $\widetilde{Y} = \widetilde{X}/\langle \widetilde{\varphi} \rangle$ then $\widetilde{Y}$ is a smooth rational surface such that the quocient map $\widetilde{h} : \widetilde{X} \to \widetilde{Y}$ is a finite morphism with degree 3, and its ramification divisor is $R = \sum_{i=1}^{9} 3D_k$.

Since, $\widetilde{h}|_{D_i} : D_i \to h(D_i)$ is a biholomorphism, the rational map $\widetilde{h}$ maps D_i in a rational curve with autointersection -3, for $i = 1, \ldots, 9$. Besides $\widetilde{h}$ maps each elliptic curve $\pi^* E_i$, $E_i \in \mathcal{E}$, in a rational curve $\widetilde{E}_i$ with autointersection -1, for $i = 1, \ldots, 12$, as showed in Figure 3.

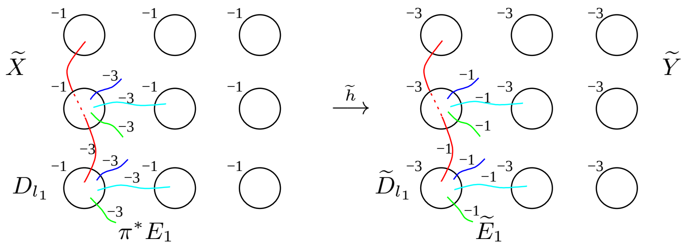

Figure 3

3.1 Relation between $\mathcal{C}$ and $\mathcal{C}^*$

Let $\pi_1 : \widetilde{Y} \to Y_0$ be the blowing-down map of the curves $\widetilde{E}_1, \ldots, \widetilde{E}_{12}$.

Lemma 3. *With the notations above defined we have that* $Y_0 = \mathbb{P}^2$.

Proof. By the Riemann-Hurwitz formula for surfaces we have

$$c_2(\widetilde{X}) = 3c_2(\widetilde{Y}) - \sum_{i=1}^{9} 2\chi(D_k),$$

where $c_2(\widetilde{X}) = 9$ and $\chi(D_k) = 2$, for $k = 1, \ldots, 9$. Therefore, $c_2(\widetilde{Y}) = 15$ and $c_2(Y_0) = 3$. This implies that Y_0 is a minimal surface, by the Noether formula (cf. [7]). Since the only minimal rational surfaces are $\mathbb{P}^2$ and the Hirzebruch surfaces S_n, with $n \geq 2$, we have $Y_0 = \mathbb{P}^2$ because $c_2(S_n) \geq 4$. $\square$

Let the rational map

$$g = \pi_1^{-1} \circ \widetilde{h} \circ \pi : X_0 \dashrightarrow Y_0 = \mathbb{P}^2$$

(see the figure 4). Let $\mathcal{E}_* := g(\mathcal{E})$ and $\mathrm{Fix}(\varphi)_* := g(\mathrm{Fix}(\varphi))$ Then g maps each elliptic curve $E \in \mathcal{E}$ in a point in $\mathbb{P}^2$, so $\mathcal{E}_*$ consist of twelve points in $\mathbb{P}^2$. Besides, g maps each $l \in \mathrm{Fix}(\varphi)$ in an algebraic

curve L in $\mathbb{P}^2$ such that $L \cdot L = 1$. In particular, L is a line in $\mathbb{P}^2$ and so, $\mathrm{Fix}(\varphi)_*$ consist of nine lines. Besides the configuration $\{\mathcal{E}_*, \mathrm{Fix}(\varphi)_*\}$ of points and lines in $\mathbb{P}^2$ satisfy the following properties

1. Each line in $\mathrm{Fix}(\varphi)_*$ contains four points of $\mathcal{E}_*$.
2. Each point of $\mathcal{E}_*$ belongs to two lines of $\mathrm{Fix}(\varphi)_*$.
3. If three points of $\mathcal{E}_*$ are not in a line in $\mathrm{Fix}(\varphi)_*$ then the points are not aligned.

Then, by Proposition 1 of [9], unless an automorphism of $\mathbb{P}^2$, we can to suppose que the configuration obtained is the configuration $\mathcal{C}$, that is, $\mathcal{C} = \big(\mathrm{Fix}(\varphi)_*, \mathcal{E}_*\big)$ that has been described in the section 2.2.

3.2 Relations between the foliations in $\mathcal{P}_4$ and $\mathcal{P}_4^*$

Recall that, fixed $\alpha \in \overline{\mathbb{C}}$, the foliation $\mathcal{G}_\alpha \in \mathcal{P}_4^*$ in X_0 is induced by the $\omega_\alpha = dy - \alpha dx$. Since the 1-form ω_α is φ-invariant, $\mathcal{G}_\alpha$ induces a foliation $\mathcal{F}_\alpha$ in $\mathbb{P}^2$ as showed in Figure 4. Besides, all the lines of

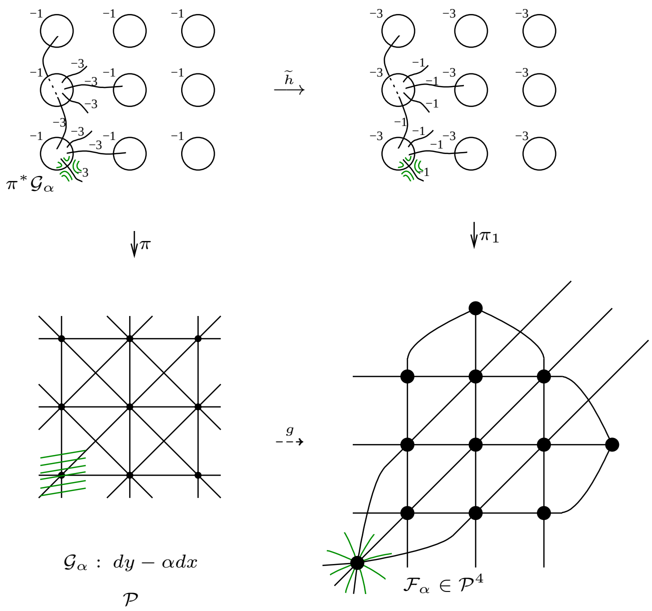

Figure 4

$\mathrm{Fix}(\varphi)_*$ are invariant respect to $\mathcal{F}_\alpha$. Then by (cf. [9, §2.2]) there exists an unique $\Lambda(\alpha) \in \overline{\mathbb{C}}$ such that $\mathcal{F}_\alpha = \mathcal{F}^4_{\Lambda(\alpha)}$, where $\mathcal{F}^4_{\Lambda(\alpha)} \in \mathcal{P}_4$. In particular $g^*(\mathcal{P}_4) = \mathcal{P}_4^*$.

Lemma 4. *The rational function $\Lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a Möbius map defined by $\Lambda(\alpha) = (\tau_0^2 - 1)\alpha + 1$.*

Proof. Since $\mathcal{F}_{\Lambda(0)}, \mathcal{F}_{\Lambda(1)}, \mathcal{F}_{\Lambda(-\tau_0)}$ and $\mathcal{F}_{\Lambda(\infty)}$ have twelve singularities, we have

$$\big\{\Lambda(0), \Lambda(1), \Lambda(-\tau_0), \Lambda(\infty)\big\} = \{1, \tau_0, \tau_0^2, \infty\}.$$

The configurations $\mathcal{C}^*$ in X and $\mathcal{C}$ in $\mathbb{P}^2$ (see Figures 1 and 2), imply

$$g^*(\mathcal{F}^4_\infty) = \mathcal{G}_\infty, \qquad g^*(\mathcal{F}^4_1) = \mathcal{G}_0,$$
$$g^*(\mathcal{F}^4_{\tau_0^2}) = \mathcal{G}_1, \qquad g^*(\mathcal{F}^4_{\tau_0}) = \mathcal{G}_{-\tau_0}.$$

Then $\Lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is an injective function such that $\Lambda(\infty) = \infty$, $\Lambda(0) = 1$, $\Lambda(1) = \tau_0^2$ and $\Lambda(-\tau_0) = \tau_0$. Therefore $\Lambda(\alpha) = (\tau_0^2 - 1)\alpha + 1 = (-2 - \tau_0)\alpha + 1$. $\square$

Remark 5. If we have a automorphism of $\mathbb{P}^2$ preserving the configuration $\mathcal{C} = \big(\mathcal{E}_*, \mathrm{Fix}(\varphi)_*\big)$ of points and lines, then Λ is a Möbius map such that

$$\big\{\Lambda(0), \Lambda(1), \Lambda(-\tau_0), \Lambda(\infty)\big\} = \{1, \tau_0, \tau_0^2, \infty\}.$$

4 Calculation of the degree of the first integral of a foliation $\mathcal{F}_t^4 \in \mathcal{P}_4$, $t \in \mathbb{Q}(\tau_0)$.

Let $\mathcal{F}_t^4 \in \mathcal{P}_4$, with $t \in \mathbb{Q}(\tau_0)$. Then there exists an unique foliation $\mathcal{G}_\alpha \in \mathcal{P}_4^*$ such that $g^*(\mathcal{G}_\alpha) = \mathcal{F}_t^4$, where $\alpha = \Lambda^{-1}(t)$. Since $\mathbb{Z}[\tau_0]$ is a unique factorization domain, we can choose $\alpha_1, \beta_1 \in \mathbb{Z}[\tau_0]$ and $(\alpha_1, \beta_1) = 1$, such that $\alpha = \dfrac{\alpha_1}{\beta_1}$. In particular, $\mathcal{G}_\alpha$ is induced by the 1-form $\omega = \beta_1 dy - \alpha_1 dx$. Besides, $f_{\alpha_1,\beta_1} = \beta_1 y - \alpha_1 x$ is a first integral of $\mathcal{G}_\alpha$ and

$$E_{\alpha_1,\beta_1} = \big\{(\alpha_1 x, \beta_1 x) : x \in E\big\}$$

is the leaf of $\mathcal{G}_\alpha$ passing by $(0, 0)$.

Let F_t be the rational first integral of $\mathcal{F}_t^4$ of degree d_t. We want to determine d_t. For this, let C a generic irreducible fiber of F_t of degree d_t. We can suppose that $C^* := g^*(C) = E_{\alpha_1,\beta_1} + p$, where $p \notin \mathrm{Fix}(\varphi)$. Let $C_{1,0}^* := E_{1,0} + p$ in X_0 and $C_{1,0} = g(C_{1,0}^*)$ the curve obtained in $\mathbb{P}^2$. The idea for calculate d_t is to find the relation between the intersection of C and $C_{1,0}$ in $\mathbb{P}^2$ and the intersection of C^* and $C_{1,0}^*$ in X_0 (see Figure 5).

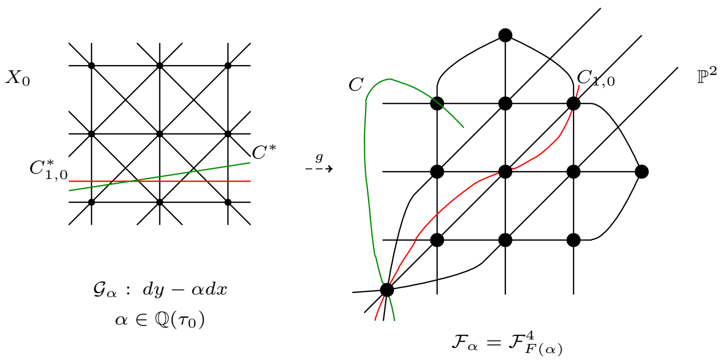

Figure 5

We observe that

$$d_t \deg(C_{1,0}) = C \cdot C_{1,0} = \pi_1^*(C) \cdot \pi_1^*(C_{1,0}). \tag{5}$$

Since $C_{1,0} \cap L_7 = \{e_{10}, e_5, e_9\}$ (see Figure 1), where e_{10}, e_5, e_9 are radial singularities of $\mathcal{F}_1^4$ and $\pi_1^* C_{1,0}$ is a regular curve, we have $\deg(C_{1,0}) = 3$. Let $\widetilde{C}$ and $\widetilde{C}_{1,0}$ the strict transformations of C and $C_{1,0}$ by π_1, respectively, then

$$\pi_1^*(C) = \widetilde{C} + \sum_{p \in \mathcal{E}_* \cap C} m_p D_p, \tag{6}$$

where m_p is the multiplicity of C in p and $D_p = \pi_1^{-1}(p)$. Besides

$$\pi_1^*(C_{1,0}) = \widetilde{C}_{1,0} + \sum_{p \in \mathcal{E}_* \cap C_{1,0}} D_p, \tag{7}$$

where $\mathcal{E}_* \cap C_{1,0} = \mathcal{E}_* \setminus \{e_1, e_6, e_8\}$.

Combining (6) and (7) in (5) we obtain

$$\begin{aligned} 3d_t &= \widetilde{C} \cdot \widetilde{C}_{1,0} + \sum_{p \in \mathcal{E}_* \cap C_{1,0}} \widetilde{C} \cdot D_p + \sum_{p \in \mathcal{E}_* \cap C} m_p \widetilde{C}_{1,0} \cdot D_p - \sum_{p \in \mathcal{E}_* \cap C_{1,0}} m_p \\ &= \widetilde{C} \cdot \widetilde{C}_{1,0} + \sum_{p \in \mathcal{E}_* \cap C_{1,0}} \widetilde{C} \cdot D_p + \sum_{p \in \mathcal{E}_* \cap C_{1,0}} m_p \widetilde{C}_{1,0} \cdot D_p - \sum_{p \in \mathcal{E}_* \cap C_{1,0}} m_p \end{aligned} \tag{8}$$

Now, given $p \in \mathcal{E}_* \cap C_{1,0}$ we have

$$\begin{aligned} \widetilde{C}_{1,0} \cdot D_p &= C^*_{1,0} \cdot E_p = 1, \\ \widetilde{C} \cdot D_p &= C^* \cdot E_p = m_p, \end{aligned}$$

where $E_p \in \mathcal{E}$ is a elliptic curve in X_0 such that $g(E_p) = p$. Hence in (8),

$$3d_t = \widetilde{C} \cdot \widetilde{C}_{1,0} + \sum_{p \in \mathcal{E}_* \cap C_{1,0}} C^* \cdot E_p. \tag{9}$$

Let $\widetilde{C^*}_{1,0}$ and $\widetilde{C^*}$, the strict transformations of $C^*_{1,0}$ and C^* by π, respectively, then $C^* \cdot C^*_{1,0} = \widetilde{C^*} \cdot \widetilde{C^*}_{1,0}$. Since $\widetilde{h}^* \widetilde{C} = 3\widetilde{C^*}$ and $\widetilde{h}^* \widetilde{C}_{1,0} = 3\widetilde{C^*}_{1,0}$ then by the Projection Formula, we have

$$3C^* \cdot 3C^*_{1,0} = 3\widetilde{C^*} \cdot 3\widetilde{C^*}_{1,0} = (\widetilde{h}^* \widetilde{C} \cdot \widetilde{h}^* \widetilde{C}_{1,0}) = 3\widetilde{C} \cdot \widetilde{C}_{1,0},$$

obtaining $3C^* \cdot C^*_{1,0} = \widetilde{C} \cdot \widetilde{C}_{1,0}$. Then, in (9), we obtain

$$\begin{aligned} 3d_t &= 3C^* \cdot C^*_{1,0} + \sum_{p \in \mathcal{E}_* \cap C^*_{1,0}} C^* \cdot E_p, \\ &= 3C^* \cdot E_{1,0} + 3C^* \cdot E_{0,1} + 3C^* \cdot E_{1,1} + 3C^* \cdot E_{1,-\tau_0}. \end{aligned}$$

Therefore

$$d_t = 3E_{\alpha_1,\beta_1} \cdot E_{1,0} + 3E_{\alpha_1,\beta_1} \cdot E_{0,1} + 3E_{\alpha_1,\beta_1} \cdot E_{1,1} + 3E_{\alpha_1,\beta_1} \cdot E_{1,-\tau_0}.$$

Denoting $N(\alpha) := N_{\mathbb{Z}[\tau_0]}(\alpha)$, $\alpha \in \mathbb{Z}(\tau_0)$, by the example 1 we have

$$3d_t = 3N(-\beta_1) + 3N(\alpha_1) + 3N(\alpha_1 - \beta_1) + 3N(-\alpha_1\tau_0 - \beta_1),$$

Hence,

$$d_t = N(\beta_1) + N(\alpha_1) + N(\beta_1 - \alpha_1) + N(\beta_1 + \tau_0\alpha_1), \tag{10}$$

where $\Lambda(t) = \alpha = \dfrac{\alpha_1}{\beta_1}$.

Remark 6. In (10) if $\alpha = a + \tau_0 b$ and $\beta = c + \tau_0 d$ then:

$$d_t = d_t(a,b,c,d) = 3(a^2 - ab + b^2 - ac + c^2 + ad - bd - cd + d^2).$$

In particular, d_t is a multiple of 3.

4.1 The growth of the pencil $\mathcal{P}_4$

In [10], Pereira defines the counting function π_C of an algebraic C curve included in $\mathbb{F}ol(2,d)$, the space of foliations in $\mathbb{P}^2$ of degree d. In this case, if $\mathcal{P} = \{\mathcal{F}_\alpha\}_{\alpha\in\overline{\mathbb{C}}}$ is a line in $\mathbb{F}ol(2,d)$, that is a pencil of foliations in $\mathbb{P}^2$, then given $n \in \mathbb{N}$, we have $\pi_{\mathcal{P}}(n) = \#E_n$, where

$$E_n = \{\alpha \in \overline{\mathbb{C}} \,:\, \mathcal{F}_\alpha \text{ have a first integral of degree at most } n\}$$

is an algebraic set of $\overline{\mathbb{C}}$.

Also, in such paper the author observes the importance of study the function $\pi_{\mathcal{P}}$ and shows the following example (cf. [10, Example 3]).

Example 7. Let $\mathcal{P} = \{\mathcal{F}_\alpha\}_{\alpha\in\overline{\mathbb{C}}}$ a pencil in $\mathbb{P}^2$, where $\mathcal{F}_\alpha$ is given by

$$\alpha x dy - y dx.$$

In this case, given $\alpha \in \overline{\mathbb{C}}$,

$$\alpha \in I_p(\mathcal{P}) \setminus \{\infty\} \Longleftrightarrow \alpha \in \mathbb{Q}.$$

Thus, suppose that $\alpha = \frac{p}{q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(p,q) = 1$. Let $f_{p,q}$ be the first integral of $\mathcal{F}_\alpha$ of degree $d_{p,q}$, then

$$d_{p,q} = \begin{cases} \max\{p,q\}, & \text{if } p \geq 0, \\ |p| + q, & \text{if } p < 0, \end{cases}$$

Then, by doing simple calculations,

$$\pi_{\mathcal{P}}(n) = 2 + 3\sum_{j=1}^{n} \varphi(j),$$

where φ is the Euler totient function. Now, since

$$\sum_{j=1}^{n} \varphi(j) = \frac{3n^2}{\pi^2} + O\Big(n \ln(n)^{2/3} \ln\big(\ln(n)\big)^{4/3}\Big),$$

(cf. [15, p. 178]), we have $\displaystyle\lim_{n\to\infty} \frac{\pi_{\mathcal{P}}(n)}{n^2} = \frac{3}{\pi^2}$.

Now, we will estimate $\pi_{\mathcal{P}^4}(n)$, for $n \in \mathbb{N}$, and see that the counting function $\pi_{\mathcal{P}^4}$ has the same behavior as in Example 7.

Corollary 8.

$$\pi_{\mathcal{P}_4}(n) = O(n^2).$$

Proof. In fact, in this case

$$t \in I_p(\mathcal{P}_4) \Longleftrightarrow \Lambda^{-1}(t) = \alpha \in \mathbb{Q}(\tau_0) \cup \{\infty\},$$

where $\Lambda(\alpha) = (\tau_0^2 - 1)\alpha + 1$. Suppose that $\alpha = \frac{\alpha_1}{\beta_1}$, $\alpha_1, \beta_1 \in \mathbb{Z}[\tau_0]$. Then

$$\pi_{\mathcal{P}_4}(n) = \#\Big\{\big(\alpha_1,\beta_1\big) \in (\mathbb{Z}[\tau_0]\times\mathbb{Z}[\tau_0]) \setminus \{0\} \,:\, (\alpha_1,\beta_1) = 1,\ d_t \leq n\Big\},$$

where $d_t = N(\beta_1) + N(\alpha_1) + N(\beta_1 - \alpha_1) + N(\beta_1 + \tau_0\alpha_1)$. Let

$$\mathcal{E}_n = \Big\{ \big(\alpha_1, \beta_1\big) \in (\mathbb{Z}[\tau_0] \times I_p(\mathcal{P})) \setminus \{0\} \ : \ t = \frac{\alpha_1}{\beta_1}, (\alpha_1, \beta_1) = 1, \ N(\alpha_1) \leq n, N(\beta_1) \leq n \Big\},$$

then

$$\pi_{\mathcal{P}_4}(n) \leq \mathcal{E}_n, \quad \forall\, n \in \mathbb{N}.$$

Let $H(n) = \big\{ I \text{ ideal in } \mathbb{Z}[\tau_0] : N_{\mathbb{Z}[\tau_0]}(I) \leq n \big\}$ then by [4] we have

1. $H(n) = cn + O(n^{1/2})$, where c is a constant.
2. $\lim\limits_{n\to\infty} \dfrac{\mathcal{E}(n)}{H(n)^2} \leq \dfrac{1}{\zeta_{\mathbb{Q}(\tau_0)}(2)}$, where $\zeta_{\mathbb{Q}(\tau_0)}$ is the Dedekind Zeta Function of $\mathbb{Q}(\tau_0)$ (see §2).

Therefore by the second item we obtain

$$\lim_{n\to\infty} \frac{\pi_{\mathcal{P}_4}(n)}{H(n)^2} \leq \frac{1}{\zeta_{\mathbb{Q}(\tau_0)}(2)}.$$

In particular $\pi_{\mathcal{P}_4}(n) = O(n^2)$. $\square$

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